2.6 曲線の長さ

問題B

1 (1) $y = x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$$L = \int_0^{\frac{4}{3}} \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$= \int_0^{\frac{4}{3}} \sqrt{1+\frac{9}{4}x}\,dx$$

$t = 1+\frac{9}{4}x$ $\quad dt = \frac{9}{4}dx$

x	$0 \to \frac{4}{3}$
t	$1 \to 4$

$$= \int_1^4 t^{\frac{1}{2}} \frac{4}{9}\,dt$$

$$= \frac{4}{9}\frac{2}{3}t^{\frac{3}{2}}\Big|_1^4 = \frac{8}{27}\left(2^3-1\right) = \frac{8}{27}\times 7 = \frac{56}{27}$$

(2) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$

$$\sqrt{1+(y')^2} = \sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}} = \sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \left(\frac{a}{x}\right)^{\frac{1}{3}}$$

したがって、全長は、

$$4\times\int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}}dx = 4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}\,dx$$

$$= 4a^{\frac{1}{3}}\frac{3}{2}x^{\frac{2}{3}}\Big|_0^a$$

$$= 6a^{\frac{1}{3}}\times a^{\frac{2}{3}} = 6a$$