4.4 逆三角関数の微分，置換積分

$\boxed{\text{問題A}}$

$\boxed{1}$ (1) $y=\cos^{-1}x$ より $\cos y = x$

$$\frac{dy}{dx}=\frac{1}{dx/dy}=\frac{1}{-\sin y}=\frac{-1}{\sqrt{1-\cos^2 y}}=\frac{-1}{\sqrt{1-x^2}}$$

(2) $y=\tan^{-1}x$ より $\tan y = x$

$$\frac{dy}{dx}=\frac{1}{dx/dy}=\frac{1}{\frac{1}{\cos^2 y}}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

$\boxed{2}$

$$\int\frac{dx}{1+x^2}=\int\frac{1}{1+\tan^2\theta}\times\frac{d\theta}{\cos^2\theta}=\int d\theta=\theta+C=\tan^{-1}x+C$$

$x=\tan\theta$

$dx=\dfrac{d\theta}{\cos^2\theta}$

$\boxed{3}$ (1)

$$\int_0^1\frac{dx}{1+x^2}=\int_0^{\frac{\pi}{4}}\frac{1}{1+\tan^2\theta}\times\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}d\theta=\Big.\theta\Big|_0^{\frac{\pi}{4}}=\frac{\pi}{4}$$

$x=\tan\theta$

$dx=\dfrac{d\theta}{\cos^2\theta}$

x	$0\to 1$
θ	$0\to\frac{\pi}{4}$

(2)

$$\int_0^1\frac{dx}{\sqrt{1-x^2}}=\int_0^{\frac{\pi}{2}}\frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}}=\int_0^{\frac{\pi}{2}}d\theta=\Big.\theta\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

$x=\sin\theta$

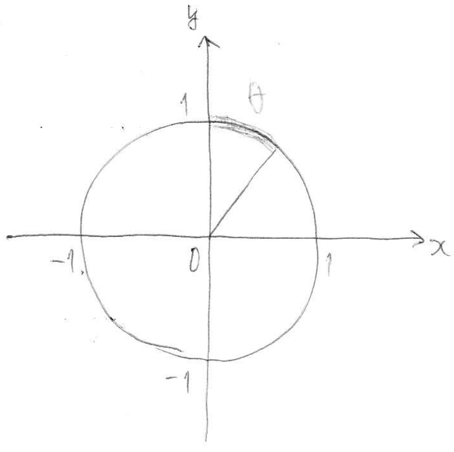
$dx=\cos\theta\,d\theta$

x	$0\to 1$
θ	$0\to\frac{\pi}{2}$

半径1の円 $x^2+y^2=1$ より $y=\sqrt{1-x^2}$

$$y'=\frac{-x}{\sqrt{1-x^2}}$$

$$1+(y')^2=1+\frac{x^2}{1-x^2}=\frac{1}{1-x^2}$$



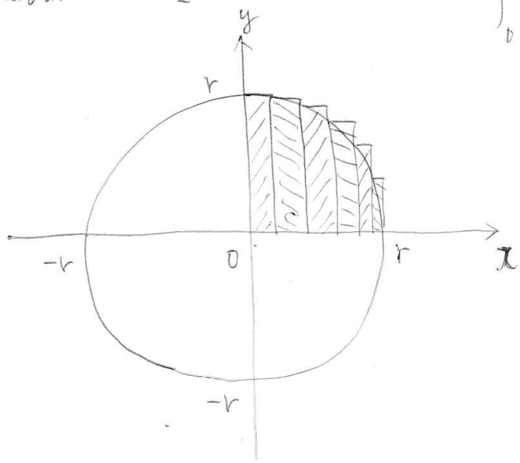
曲線の長さ $\theta=\displaystyle\int_0^1\sqrt{1+(y')^2}\,dx=\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx$

$\boxed{4}$ $\displaystyle\int_0^r \sqrt{r^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{r^2-r^2\sin^2\theta}\; r\cos\theta\,d\theta = r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$x = r\sin\theta$ $\quad \begin{array}{c|c} x & 0\to r \\ \hline \theta & 0\to\frac{\pi}{2} \end{array}$

$dx = r\cos\theta\,d\theta$

$$= r^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{r^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi r^2}{4}$$



問題B

$\boxed{1}$ (1) $\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$

$x = \sin\theta$ $\quad \begin{array}{c|c} x & 0\to 1 \\ \hline \theta & 0\to\frac{\pi}{2} \end{array}$

$dx = \cos\theta\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{16}$$