2.2 Rotation of the probability and expectation value

§1. Rotation in z-axis

z軸に関して角度φの回転に対して、状態$|\alpha\rangle$は、

$$|\alpha\rangle \longrightarrow |\alpha\rangle_R = \mathcal{D}_z(\varphi)|\alpha\rangle$$

となる。この回転によって期待値は、例えばS_xをとると、

$$\langle S_x\rangle \longrightarrow \langle S_x\rangle_R = {}_R\langle\alpha|S_x|\alpha\rangle_R = \langle\alpha|\mathcal{D}_z^\dagger(\varphi)\, S_x\, \mathcal{D}_z(\varphi)|\alpha\rangle$$

のように変換される。

いま、$|\alpha\rangle = |+\rangle$とすると、$|+\rangle_R = \mathcal{D}_z(\varphi)|+\rangle$

① 回転前と同じ状態をとる確率

$$\langle +|+\rangle_R = \langle +|\mathcal{D}_z(\varphi)|+\rangle = (1,0)\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (1,0)\begin{pmatrix}e^{-i\varphi/2}\\0\end{pmatrix} = e^{-i\varphi/2}$$

$$|\langle +|+\rangle_R|^2 = |e^{-i\varphi/2}|^2 = 1$$

$$\langle -|+\rangle_R = \langle -|\mathcal{D}_z(\varphi)|+\rangle = (0,1)\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (0,1)\begin{pmatrix}e^{-i\varphi/2}\\0\end{pmatrix} = 0$$

$$|\langle -|+\rangle_R|^2 = 0$$

② 期待値

$$\langle S_x \rangle_R = \langle + | \underline{\mathcal{D}_z^{+} S_x \mathcal{D}_z} | + \rangle$$

$$= \frac{\hbar}{2}(1,0)\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1,0)\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 0 & e^{i\varphi/2} \\ e^{-i\varphi/2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1,0)\underline{\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1,0)\begin{pmatrix} 0 \\ e^{-i\varphi} \end{pmatrix} = 0$$

$$\langle S_y \rangle_R = \langle + | \underline{\mathcal{D}_z^{+} S_y \mathcal{D}_z} | + \rangle$$

$$= \frac{\hbar}{2}(1,0)\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1,0)\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 0 & -ie^{i\varphi/2} \\ ie^{-i\varphi/2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1,0)\underline{\underline{\begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix}}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1,0)\begin{pmatrix} 0 \\ ie^{-i\varphi} \end{pmatrix} = 0$$

$$\langle S_z \rangle_R = \langle + | \underline{\mathcal{D}_z^{+} S_z \mathcal{D}_z} | + \rangle$$

$$= \frac{\hbar}{2}(1,0)\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1,0)\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & -e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1,0)\underline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1,0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{2}$$

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量子力学 No.6 Rotation in z-axis

1. z 軸に関する回転前の状態が

$$|\alpha\rangle = |x+\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{1}$$

であるとき，次の量を求めなさい．

(a) $\langle x+|\mathcal{D}_z|x+\rangle$ および $|\langle x+|\mathcal{D}_z|x+\rangle|^2$

$$\frac{1}{2}(1,1)\begin{pmatrix}e^{-i\varphi/2} & 0\\ 0 & e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(1,1)\begin{pmatrix}e^{-i\varphi/2}\\ e^{i\varphi/2}\end{pmatrix} = \frac{1}{2}\left(e^{-i\varphi/2}+e^{i\varphi/2}\right) = \cos\frac{\varphi}{2}$$

$$|\langle x+|\mathcal{D}_z|x+\rangle|^2 = \cos^2\frac{\varphi}{2}$$

(b) $\langle x-|\mathcal{D}_z|x+\rangle$ および $|\langle x-|\mathcal{D}_z|x+\rangle|^2$

$$\frac{1}{2}(1,-1)\begin{pmatrix}e^{-i\varphi/2} & 0\\ 0 & e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(1,-1)\begin{pmatrix}e^{-i\varphi/2}\\ e^{i\varphi/2}\end{pmatrix} = \frac{1}{2}\left(e^{-i\varphi/2}-e^{i\varphi/2}\right) = -i\sin\frac{\varphi}{2}$$

$$|\langle x-|\mathcal{D}_z|x+\rangle|^2 = \left|-i\sin\frac{\varphi}{2}\right|^2 = \sin^2\frac{\varphi}{2}$$

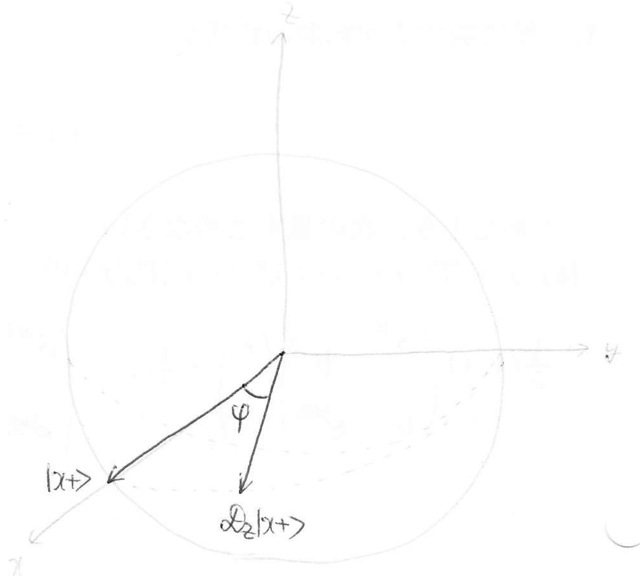
2. 表面のとき，次の期待値を求めなさい．

(a) $\langle S_x \rangle \to \langle S_x \rangle_R = \langle x+|\mathcal{D}_z^\dagger S_x \mathcal{D}_z|x+\rangle$

$$= \frac{1}{2}\frac{\hbar}{2}(1,1)\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}(1,1)\begin{pmatrix} e^{-i\varphi} \\ e^{i\varphi} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{-i\varphi} + e^{i\varphi}\right) = \frac{\hbar}{2}\cos\varphi$$



(b) $\langle S_y \rangle \to \langle S_y \rangle_R = \langle x+|\mathcal{D}_z^\dagger S_y \mathcal{D}_z|x+\rangle$

$$= \frac{1}{2}\frac{\hbar}{2}(1,1)\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}(1,1)\begin{pmatrix} -ie^{-i\varphi} \\ ie^{i\varphi} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(-ie^{-i\varphi} + ie^{i\varphi}\right) = \frac{\hbar}{2}\frac{e^{i\varphi} - e^{-i\varphi}}{2i} = \frac{\hbar}{2}\sin\varphi$$

(c) $\langle S_z \rangle \to \langle S_z \rangle_R = \langle x+|\mathcal{D}_z^\dagger S_z \mathcal{D}_z|x+\rangle$

$$= \frac{1}{2}\frac{\hbar}{2}(1,1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}(1,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\times 0 = 0$$

3. 今日の講義でわかったこと・わからなかったこと・感想などを書きなさい．(自由記載)

番号　　　　　　　　　　氏名

量子力学 No.6-1 Rotation in z-axis

1. z 軸に関する回転前の状態が

$$|\alpha\rangle = |y+\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \tag{2}$$

であるとき，次の量を求めなさい．

(a) $\langle y+|\mathcal{D}_z|y+\rangle$ および $|\langle y+|\mathcal{D}_z|y+\rangle|^2$

$$\frac{1}{2}(1,-i)\begin{pmatrix}e^{-i\varphi/2} & 0\\ 0 & e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}(1,-i)\begin{pmatrix}e^{-i\varphi/2}\\ ie^{i\varphi/2}\end{pmatrix} = \frac{1}{2}\left(e^{-i\varphi/2}+e^{i\varphi/2}\right) = \cos\frac{\varphi}{2}$$

$$|\langle y+|\mathcal{D}_z|y+\rangle|^2 = \cos^2\frac{\varphi}{2}$$

(b) $\langle y-|\mathcal{D}_z|y+\rangle$ および $|\langle y-|\mathcal{D}_z|y+\rangle|^2$

$$\frac{1}{2}(1,i)\begin{pmatrix}e^{-i\varphi/2} & 0\\ 0 & e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}(1,i)\begin{pmatrix}e^{-i\varphi/2}\\ ie^{i\varphi/2}\end{pmatrix} = \frac{1}{2}\left(e^{-i\varphi/2}-e^{i\varphi/2}\right) = -i\sin\frac{\varphi}{2}$$

$$|\langle y-|\mathcal{D}_z|y+\rangle|^2 = \left|-i\sin\frac{\varphi}{2}\right|^2 = \sin^2\frac{\varphi}{2}$$

2. 表面のとき，次の期待値を求めなさい．

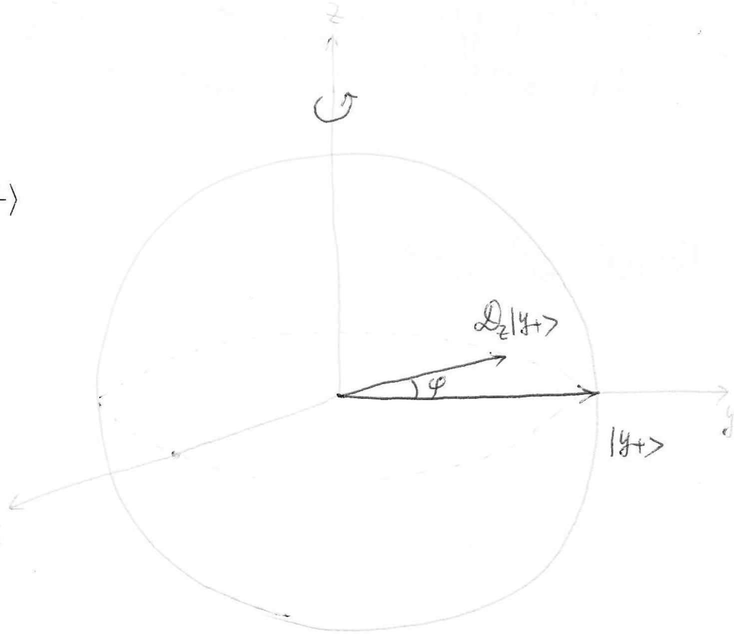
(a) $\langle S_x \rangle \to \langle S_x \rangle_R = \langle y + |\mathcal{D}_z^\dagger S_x \mathcal{D}_z| y+\rangle$

$$= \frac{1}{2}\frac{\hbar}{2}(1,-i)\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{\hbar}{4}(1,-i)\begin{pmatrix} i e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(i e^{i\varphi} - i e^{-i\varphi}\right)$$

$$= \frac{\hbar}{2}\, ii\, \frac{e^{i\varphi} - e^{-i\varphi}}{2i} = -\frac{\hbar}{2}\sin\varphi$$



(b) $\langle S_y \rangle \to \langle S_y \rangle_R = \langle y + |\mathcal{D}_z^\dagger S_y \mathcal{D}_z| y+\rangle$

$$= \frac{1}{2}\frac{\hbar}{2}(1,-i)\begin{pmatrix} 0 & -i e^{i\varphi} \\ i e^{-i\varphi} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{\hbar}{4}(1,-i)\begin{pmatrix} e^{i\varphi} \\ i e^{-i\varphi} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\varphi} + e^{-i\varphi}\right)$$

$$= \frac{\hbar}{2}\cos\varphi$$

(c) $\langle S_z \rangle \to \langle S_z \rangle_R = \langle y + |\mathcal{D}_z^\dagger S_z \mathcal{D}_z| y+\rangle$

$$= \frac{1}{2}\frac{\hbar}{2}(1,-i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{\hbar}{4}(1,-i)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \frac{\hbar}{4}(1-1)$$

$$= 0$$

ちなみに、

$$\mathcal{D}_z^{\dagger} S_x \mathcal{D}_z = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\varphi + i\sin\varphi \\ \cos\varphi - i\sin\varphi & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos\varphi - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi$$

$$= S_x \cos\varphi - S_y \sin\varphi$$

$$\mathcal{D}_z^{\dagger} S_y \mathcal{D}_z = \frac{\hbar}{2}\begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\cos\varphi + \sin\varphi \\ i\cos\varphi + \sin\varphi & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\varphi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\varphi = S_x \sin\varphi + S_y \cos\varphi$$

$$\mathcal{D}_z^{\dagger} S_z \mathcal{D}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S_z$$