番号　　　　　　　　氏名

特殊相対論 No.13 運動方程式を解く (2) 鉛直投げ上げ (動力学 No.8 参照)

$$\begin{cases} x_r(t+\epsilon) &= x_r(t) + \epsilon v_r\left(t+\dfrac{\epsilon}{2}\right) \\ v_r\left(t+\dfrac{\epsilon}{2}\right) &= v_r\left(t-\dfrac{\epsilon}{2}\right) - \epsilon g\left\{1-\left(v_r\left(t-\dfrac{\epsilon}{2}\right)/c\right)^2\right\}^{3/2} \end{cases}
\qquad
\begin{cases} x_c(t+\epsilon) &= x_c(t) + \epsilon v_c\left(t+\dfrac{\epsilon}{2}\right) \\ v_c\left(t+\dfrac{\epsilon}{2}\right) &= v_c\left(t-\dfrac{\epsilon}{2}\right) - \epsilon g \end{cases}$$

ここで，$\epsilon = 0.20$ s，$g = 1.0$ m/s^2，$c = 1.0$ m/s とし，小数第 4 位を四捨五入しなさい．

時刻 t	位置 $x_r(t)$	速さ $v_r(t)$	位置 $x_c(t)$	速さ $v_c(t)$
0	$x_r(0) = 0.0$	$v_r(0) = 0.6$	$x_c(0) = 0.0$	$v_c(0) = 0.6$
		$v_r(\frac{\epsilon}{2}) = v_r(0) - \frac{\epsilon}{2}g\left\{1-\left(\frac{v_r(0)}{c}\right)^2\right\}^{3/2}$		$v_c(\frac{\epsilon}{2}) = v_c(0) - \frac{\epsilon}{2}g$
ϵ	0.109	$= 0.545$	0.100	$= 0.500$
		0.410		0.300
2ϵ	0.191		0.160	
		0.243		0.100
3ϵ	0.239		0.180	
		0.050		−0.100
4ϵ	0.249		0.160	
		−0.148		−0.300
5ϵ	0.219		0.100	
		−0.330		−0.500
6ϵ	0.153		0.000	
		−0.482		−0.700
7ϵ	0.057		−0.140	
		−0.600		−0.900
8ϵ	−0.062		−0.320	
		−0.689		−1.100
9ϵ	−0.200		−0.540	
		−0.755		−1.300
10ϵ	−0.351		−0.800	
		−0.804		−1.500
11ϵ	−0.511		−1.100	
		−0.840		−1.700
12ϵ	−0.679		−1.440	
		−0.868		−1.900
13ϵ	−0.853		−1.820	
		−0.890		−2.100
14ϵ	−1.031		−2.240	
		−0.907		−2.300
15ϵ	−1.212		−2.700	
		*****		*****

1. $v-t$ グラフ，$x-t$ グラフをグラフ用紙に描きなさい．

2. 質量 m の物体に一定の力 $F=-mg$ が働くときの運動方程式を解こう.

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-(v/c)^2}}\right)=F=-mg \tag{1}$$

(a) 初期条件 ($t=0$ のとき $v_0=+0.6c$) のとき t で積分して

$$\frac{v_r}{c}=\frac{\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}-\frac{gt}{c}}{\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}-\frac{gt}{c}\right)^2}} \tag{2}$$

となることを示しなさい.

(b) 式 (2) について, $t\to\infty$ の極限を求めなさい.

(c) 式 (2) について, $c\to\infty$ の極限を求めなさい. (Newton 力学の解と一致するか?)

3. $v=\frac{dx}{dt}$ より, 初期条件 ($t=0$ のとき $x_0=0$) のもとで (2) 式を t について積分して

$$x_r=\frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}}-\frac{c^2}{g}\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}-\frac{gt}{c}\right)^2} \tag{3}$$

となることを示しなさい.

(a) 式 (3) より漸近線を求め, 問 **1.** で描いた $x-t$ グラフに漸近線を書き入れなさい.

(b) 式 (3) について, $c\to\infty$ の極限を求めなさい. (Newton 力学の解と一致するか?)

4. 上に描いたグラフより次の座標を求めなさい.

(a) w 軸との交点 A の座標 w_A を求めなさい.

(b) 地球から最も遠ざかった点 B の時間座標は $2w_B=w_A$ であることから, x_B を求めなさい.

5. 式 (2) から物体の固有時 τ

$$\tau = \int_0^t dt \sqrt{1-(v_r/c)^2} \tag{4}$$

を計算し，$w = ct$ と x を τ で表しなさい．

Hint : $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x$

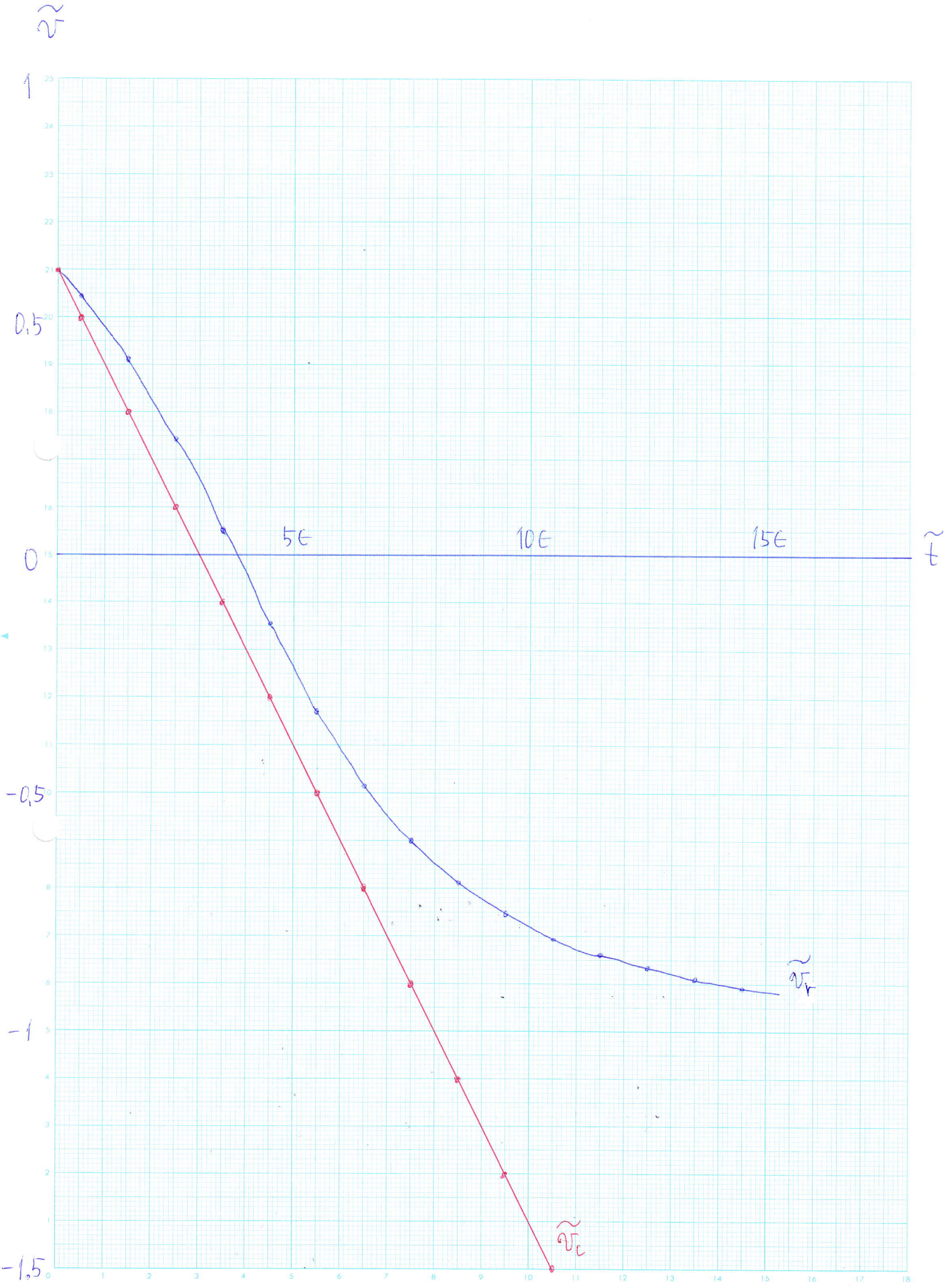
6. x_B のときの物体の固有時を τ_B とする．物体の座標 (w, x) を τ_B を使ってあらわすと

$$w = \frac{c^2}{g}\left[\sinh\left(\frac{g}{c}\tau_B\right) + \sinh\left(\frac{g}{c}\tau - \frac{g}{c}\tau_B\right)\right] \tag{5}$$

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g}{c}\tau_B\right) - \cosh\left(\frac{g}{c}\tau - \frac{g}{c}\tau_B\right)\right] \tag{6}$$

となることを示しなさい．

7. 物体が地球に戻る時刻は，物体の固有時で $\tau_A = 2\tau_B$ である．w_A と $c\tau_A$ の大小関係を決定しなさい．



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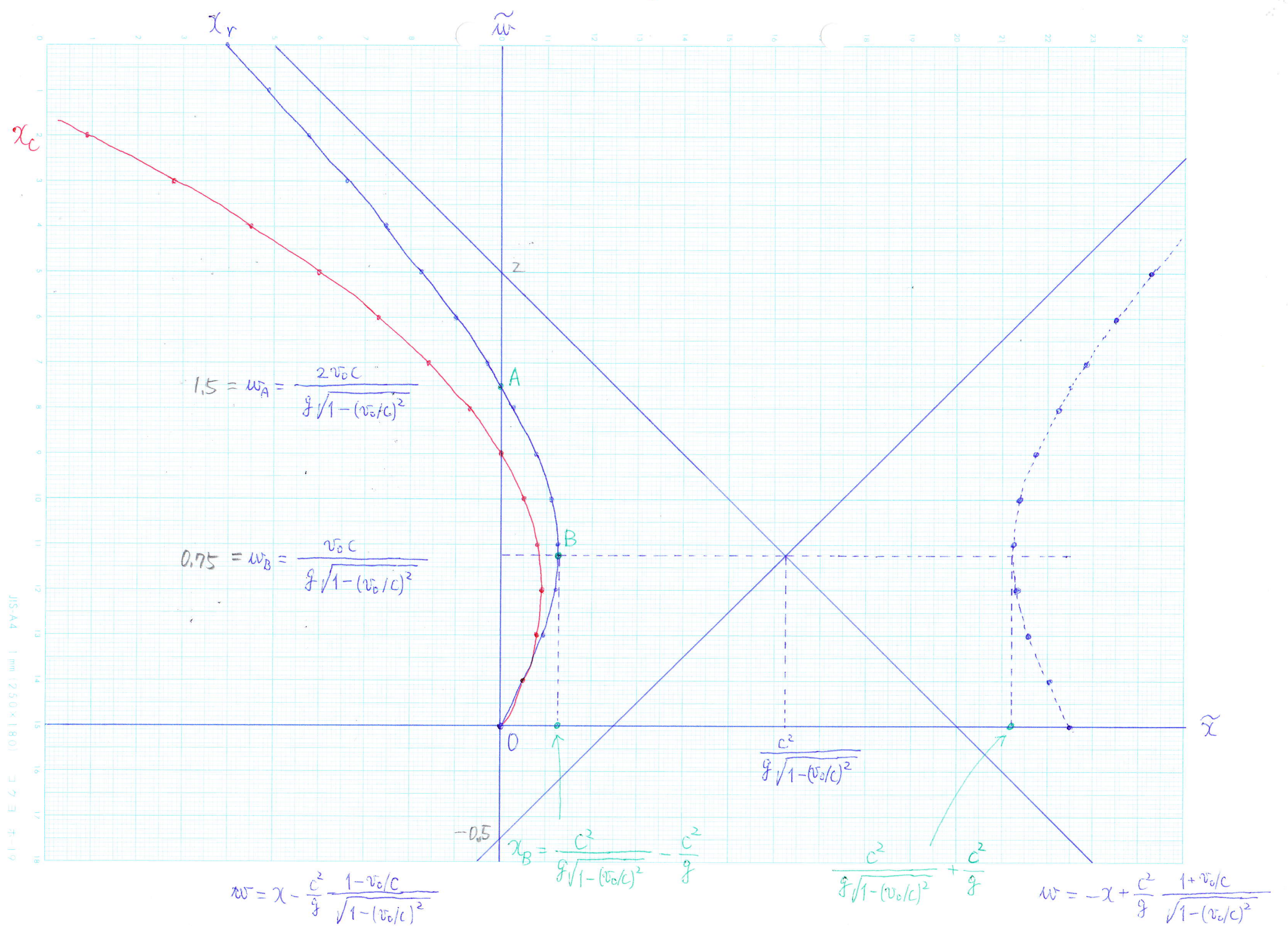


x_r
$\tilde{w}$
x_c
2
A
$1.5 = w_A = \frac{2 v_0 c}{g\sqrt{1-(v_0/c)^2}}$
B
$0.75 = w_B = \frac{v_0 c}{g\sqrt{1-(v_0/c)^2}}$
O
$\tilde{x}$
$\frac{c^2}{g\sqrt{1-(v_0/c)^2}}$
-0.5
$x_B = \frac{c^2}{g\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}$
$\frac{c^2}{g\sqrt{1-(v_0/c)^2}} + \frac{c^2}{g}$
$w = x - \frac{c^2}{g}\frac{1-v_0/c}{\sqrt{1-(v_0/c)^2}}$
$w = -x + \frac{c^2}{g}\frac{1+v_0/c}{\sqrt{1-(v_0/c)^2}}$

$\boxed{2}$ (a) $$\frac{mv}{\sqrt{1-(v/c)^2}} = -mgt + A\ (\text{積分定数})$$

初期条件より $$\frac{mv_0}{\sqrt{1-(v_0/c)^2}} = A$$

したがって

$$\frac{mv}{\sqrt{1-(v/c)^2}} = A - mgt$$

$$\frac{m^2v^2}{1-(v/c)^2} = (A-mgt)^2$$

$$m^2v^2 = (A-mgt)^2 - \left(\frac{A-mgt}{c}\right)^2 v^2$$

$$\left\{m^2 + \left(\frac{A-mgt}{c}\right)^2\right\} v^2 = (A-mgt)^2$$

正符号を取る。初期条件に合うため。

$$\therefore v = \frac{A-mgt}{\sqrt{m^2+\left(\frac{A-mgt}{c}\right)^2}} = \frac{\frac{mv_0}{\sqrt{1-(v_0/c)^2}} - mgt}{\sqrt{m^2+\left(\frac{mv_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{mgt}{c}\right)^2}}$$

$$= \frac{\frac{v_0}{\sqrt{1-(v_0/c)^2}} - gt}{\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}}$$

(b)

$$v = \frac{\frac{1}{t}\frac{v_0}{\sqrt{1-(v_0/c)^2}} - g}{\sqrt{\frac{1}{t^2}+\left(\frac{1}{t}\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c}\right)^2}} \longrightarrow \frac{-g}{g/c} = -c \quad (t\to\infty)$$

(c)

$$v = \frac{\frac{v_0}{\sqrt{1-(v_0/c)^2}} - gt}{\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}} \longrightarrow v_0 - gt \quad (c\to\infty)$$

3

$$\frac{x}{c} = \int dt \frac{v}{c} = \int dt \frac{\dfrac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \dfrac{gt}{c}}{\sqrt{1+\left(\dfrac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \dfrac{gt}{c}\right)^2}}$$

$$= -\frac{c}{g}\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2} + B(\text{積分定数})$$

初期条件より

$$B = \frac{c}{g}\sqrt{1+\frac{(v_0/c)^2}{1-(v_0/c)^2}} = \frac{c}{g}\sqrt{\frac{1}{1-(v_0/c)^2}} = \frac{c}{g}\frac{1}{\sqrt{1-(v_0/c)^2}}$$

したがって、

$$\boxed{x = \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}}$$

(a)

$$\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2} = \frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c^2}x$$

$$1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2 = \left(\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c^2}x\right)^2$$

$$\left(\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c^2}x\right)^2 - \left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2 = 1$$

$$\underbrace{\left(\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c^2}x + \frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)}\underbrace{\left(\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c^2}x - \frac{v_0/c}{\sqrt{1-(v_0/c)^2}} + \frac{gt}{c}\right)} = 1$$

$$\frac{gw}{c^2} = -\frac{g}{c^2}x + \frac{1+v_0/c}{\sqrt{1-(v_0/c)^2}} \qquad \frac{gw}{c^2} = \frac{g}{c^2}x - \frac{1-v_0/c}{\sqrt{1-(v_0/c)^2}}$$

$$w = -x + \frac{c^2}{g}\frac{1+v_0/c}{\sqrt{1-(v_0/c)^2}} \qquad w = x - \frac{c^2}{g}\frac{1-v_0/c}{\sqrt{1-(v_0/c)^2}}$$

$g = c = 1$, $\dfrac{v_0}{c} = 0.6 = \dfrac{6}{10} = \dfrac{3}{5}$ のとき

$$w = -x + 2 \qquad w = x - \frac{1}{2}$$

$\boxed{3}$ (b)

$$x = \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}$$

$$= \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\left\{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2\right\}^{\frac{1}{2}}$$

$$\fallingdotseq \frac{c^2}{g} - \frac{c^2}{g}\left\{1+\frac{1}{2}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2\right\}$$

$$= \frac{c^2}{g} - \frac{c^2}{g} - \frac{c^2}{2g}\left(\frac{(v_0/c)^2}{1-(v_0/c)^2} - 2\frac{gt}{c}\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} + \frac{g^2t^2}{c^2}\right)$$

$$= \frac{c^2}{g} - \frac{c^2}{g} - \frac{c^2}{2g}\left(0 - 2\frac{gt}{c}\frac{v_0}{c} + \frac{g^2t^2}{c^2}\right)$$

$$= + v_0 t - \frac{g}{2}t^2$$

4 (a) $x=0$より

$$\frac{1}{1-(v_0/c)^2} = 1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2$$

$$\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2 = \frac{1}{1-(v_0/c)^2} - 1 = \frac{(v_0/c)^2}{1-(v_0/c)^2}$$

$$\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c} = \pm\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}$$

$$\frac{gt}{c} = \frac{v_0/c}{\sqrt{1-(v_0/c)^2}} \pm \frac{v_0/c}{\sqrt{1-(v_0/c)^2}} = 0,\ \frac{2v_0/c}{\sqrt{1-(v_0/c)^2}}$$

したがって.

$$w_A = \frac{c^2}{g}\frac{2v_0/c}{\sqrt{1-(v_0/c)^2}}$$

$v_0=0.6c$のとき $w_A = \frac{3}{2} = 1.5$ $w_B = \frac{3}{4} = 0.75$

(b)

$$x_B = \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\sqrt{1+\Big(\underbrace{\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{g}{c}\frac{c}{g}\times\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}}_{=0}\Big)^2}$$

$$= \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}$$

$v_0=0.6c$のとき $x_B = \frac{1}{4} = 0.25$

5

$$1-\frac{v^2}{c^2} = 1 - \frac{\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2} = \frac{1}{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}$$

したがって,

$$\tau = \int_0^t dt \frac{1}{\sqrt{1+\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)^2}}$$

$$= -\frac{c}{g} \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right)\Bigg|_0^t$$

$$= -\frac{c}{g} \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right) + \frac{c}{g} \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)$$

$$\frac{c}{g} \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c}\right) = -\tau + \frac{c}{g} \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)$$

$$\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{gt}{c} = \sinh\left(-\frac{g}{c}\tau + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)\right)$$

$$t = \frac{c}{g}\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{c}{g}\sinh\left(-\frac{g}{c}\tau + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)\right)$$

$$\boxed{w = \frac{c^2}{g}\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\sinh\left(-\frac{g}{c}\tau + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)\right)}$$

$$x = \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\sqrt{1+\sinh^2\left(-\frac{g}{c}\tau + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)\right)}$$

$$\therefore \boxed{x = \frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\cosh\left(-\frac{g}{c}\tau + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)\right)}$$

6

$$x_B = \underbrace{\frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}}_{\boxed{4}(b)} = \underbrace{\frac{c^2}{g}\frac{1}{\sqrt{1-(v_0/c)^2}} - \frac{c^2}{g}\cosh\left(-\frac{g}{c}\tau_B + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)\right)}_{\boxed{5}}$$

より

$$\cosh\Bigg(\underbrace{-\frac{g}{c}\tau_B + \operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right)}_{=0}\Bigg) = 1$$

より $\operatorname{arcsinh}\left(\frac{v_0/c}{\sqrt{1-(v_0/c)^2}}\right) = \frac{g}{c}\tau_B$

$$\frac{v_0/c}{\sqrt{1-(v_0/c)^2}} = \sinh\left(\frac{g}{c}\tau_B\right) = \frac{3}{4}\ (v=0.6c\text{のとき})$$

また、

$$\cosh\left(\frac{g}{c}\tau_B\right) = \sqrt{1+\sinh^2\left(\frac{g}{c}\tau_B\right)} = \sqrt{1+\frac{(v_0/c)^2}{1-(v_0/c)^2}} = \frac{1}{\sqrt{1-(v_0/c)^2}} = \frac{5}{4}\ (v=0.6c\text{のとき})$$

以上から、

$$w = \frac{c^2}{g}\sinh\left(\frac{g}{c}\tau_B\right) - \frac{c^2}{g}\sinh\left(-\frac{g}{c}\tau + \frac{g}{c}\tau_B\right) = \frac{c^2}{g}\sinh\left(\frac{g}{c}\tau_B\right) + \frac{c^2}{g}\sinh\left(\frac{g}{c}\tau - \frac{g}{c}\tau_B\right) \quad \text{---- (5)}$$

$$x = \frac{c^2}{g}\cosh\left(\frac{g}{c}\tau_B\right) - \frac{c^2}{g}\cosh\left(-\frac{g}{c}\tau + \frac{g}{c}\tau_B\right) = \frac{c^2}{g}\cosh\left(\frac{g}{c}\tau_B\right) - \frac{c^2}{g}\cosh\left(\frac{g}{c}\tau - \frac{g}{c}\tau_B\right) \quad \text{--- (6)}$$

7 式(5)に $\tau = \tau_A = 2\tau_B$ を代入すると、

$$w_A = \frac{c^2}{g}\sinh\left(\frac{g}{c}\tau_B\right) + \frac{c^2}{g}\sinh\left(\frac{g}{c}\tau_B\right) = \frac{2c^2}{g}\sinh\left(\frac{g}{c}\tau_B\right)$$

したがって、

$$\frac{w_A}{c\tau_A} = \frac{\frac{2c^2}{g}\sinh\left(\frac{g}{c}\tau_B\right)}{2c\tau_B} = \frac{\sinh\left(\frac{g}{c}\tau_B\right)}{\frac{g}{c}\tau_B} > 1 \text{ より } w_A > c\tau_A$$

番号　　　　　　　　氏名

特殊相対論 No.13-2　ふたごのパラドックス

1. ロケット君は時刻 $t=0$ に速さ $+0.6c$ で地球を出発し，$x=3$ m の点 B で向きを逆にし，速さ $-0.6c$ で地球に戻ってくる．地球に戻ったときを A 点とする．

(a) この時空図を描きなさい．

(b) ロケット君が地球に戻ってきたとき，地上君の時刻 w_A は何光秒か．

$w_A = +10$ 光秒

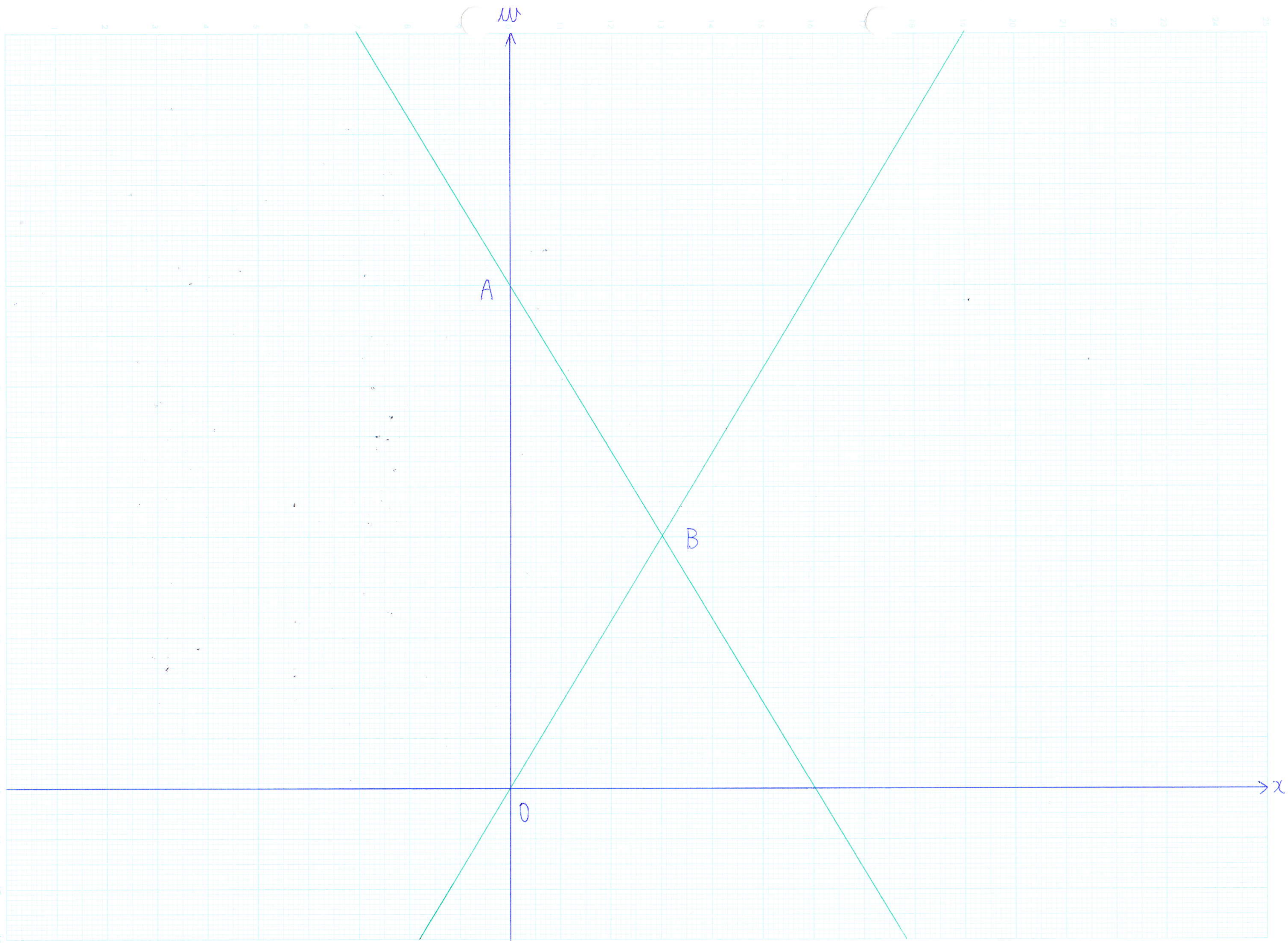
(c) ロケット君の世界線の長さ（ロケット君の固有時）$c\tau_A$ は何光秒か．

$\beta = 0.6$ のとき $\alpha_r = \sqrt{\dfrac{1+\beta^2}{1-\beta^2}} = 1.46$ を使う．

$c\tau_A = 2 \times \dfrac{5.83}{\alpha_r} = 2 \times \dfrac{5.83}{1.46} = 7.99$ 光秒

2. ふたごのパラドックスは，パラドックスか？

3. 今日の講義でわかったこと・わからなかったこと・感想などを書きなさい．(自由記載)


w
A
B
x
O